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Random initial conditions and nonlinear relaxation

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Abstract. We study the effect of randomness in the initial conditions on the deterministic diffusion equation with nonlinear terms. Physically, this describes, among other things, the time development of a system quenched from a high temperature to the vicinity of the critical point, in the approximation where the effects of thermal noise are neglected. We consider the case of a non-conserved order parameter with $O(n)$ symmetry, and show that the nonlinearities are irrelevant for the large time behaviour for dimension $d > 2$. The model is investigated for $d < 2$ using the renormalization group and ϵ-expansion. It is found, to all orders in ϵ, that the local fluctuations in the order parameter scale like $t^{-1/2}$, and have a universal distribution. The time dependence of the response function, describing the dependence on the initial condition, is characterised by another exponent which is computed to $O(\epsilon^2)$. These results are checked in the exactly soluble cases of $n \to \infty$ and $d = 0$.

1. Introduction

Nonlinear relaxation processes are found in many areas of physics, chemistry and biology. In the generic case, the effect of the nonlinearities is qualitatively unimportant for the large time behaviour, which exhibits the usual kind of exponential decay characterized by a finite relaxation time scale. Under such circumstances, any randomness present in the initial conditions tends also to be suppressed exponentially, and leads to no qualitative differences in the large time behaviour. However, when the nonlinearities dominate, the exponential behaviour is typically modified to that of a power law. In such circumstances, as we shall show, initial state randomness may be very important, and lead to strong qualitative differences in the long time properties. The manner in which this comes about is analogous in many ways to the behaviour of critical fluctuations close to a second-order phase transition, and the methods we use in this paper follow closely those of the renormalization group so successfully applied in the latter class of problems.

For the case of a system described by a single space and time-dependent field $\phi(x,t)$ (referred to, by analogy with critical phenomena, as the order parameter), we consider nonlinear relaxation processes governed by a deterministic equation of the form

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta F}{\delta \phi} \tag{1}$$

where $F\{\phi\}$ is some functional, which, by the form of the equation, is non-increasing as a function of time. Throughout most of this paper, we consider the following specific form for F

$$F = \int \left[\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}r\phi^2 + \tfrac{1}{4}\lambda\phi^4\right] \mathrm{d}^d x \tag{2}$$

although, as will be shown, many of the results are universal and do not depend on the detailed form. When $r \neq 0$, the long time relaxation is exponential, with a characteristic time $\sim r^{-1}$, but when $r = 0$, this is replaced by a power law. This is the analogue of the critical point. Even when r is small, we expect the nonlinearities to dominate intermediate times and lead to effective power laws in that regime. Equation (1) is completely deterministic and cannot by itself lead to the kind of non-trivial power law decays associated with true critical phenomena. However, we also suppose that the initial condition $\phi(x, 0)$ is a random variable, with some strength characterized by a parameter Δ. For example, it could have a white noise distribution with $\overline{\phi(x,0)\phi(x',0)} = \Delta\delta(x - x')$. Once again, we shall show that universality implies that the precise details of this distribution are not important. Then it will turn out that this system, although deterministic for $t > 0$, can exhibit non-trivial critical behaviour at $r = 0$.

The only stochastic aspect of the problem so far considered is the randomness in the initial condition. When a random noise term $\eta(x, t)$ is added to the right-hand side of equation (1), it becomes the usual time-dependent Landau–Ginzburg equation used to describe dynamic critical phenomena [1, 2]. In that case, F is simply the coarse-grained free energy (in units of $k_{\mathrm{B}}T$), and the noise η represents the effect of thermal fluctuations on scales smaller than that of the coarse-graining. It is usually chosen also to have a white noise distribution, satisfying $\overline{\eta(x,t)\eta(x',t')} = 2D\delta(x - x')\delta(t - t')$. The principle of detailed balance applied to small fluctuations in the equilibrium state then implies that $D = \Gamma$. When the thermal noise is present, we would expect it to wash out the effects of the randomness in the initial conditions on a relatively short time scale. For example, if this randomness results from a quench from a relatively high temperature to the vicinity of the critical point, one may show (see appendix A) that $\Delta \sim R^2$ where R is the range of the interaction. Thus the ratio Δ/D of the initial noise to the thermal noise gives rise to a characteristic time $\sim R^2/D$. When detailed balance is satisfied, this becomes R^2/Γ. This is typically a microscopic time. It is to be compared with the relaxation time close to T_{c}, which is $\sim \xi^2/\Gamma$, where the correlation length ξ is always larger than R, and diverges at T_{c}.

The relative unimportance of the initial state randomness for equilibrium phenomena appears in the renormalization group approach as a consequence of the parameter Δ being strongly irrelevant at the fixed point, with $D \neq 0$, which describes conventional critical dynamics. However, as shown by Janssen *et al* [3] and by Humayun and Bray [4], this irrelevant operator leads to a new non-trivial exponent governing the response function describing the dependence on the initial conditions of quantities at large times. (See also Huse [5].) This is not the case studied in this paper. Instead, we are interested in the fixed point where $D = 0$.

In order to realize such a situation, it is clearly necessary to consider systems in which detailed balance is violated, and the effects of thermal noise are negligible. There are many systems, for example those maintained in a steady non-equilibrium state by some external driving force, for which this can be a reasonable approximation, and we expect our analysis to be applicable in such cases.

Even when the effects of initial noise are important, there remains the question of how they are affected by the nonlinearities in the system. For the thermal fluctuations, this question is answered by the Ginzburg criterion, which implies, for an ordinary critical point, that for dimension $d > 4$ such nonlinearities are irrelevant, while for $d < 4$ they are important sufficiently close to the critical point. We shall show in this paper that, for the case of a non-conserved order parameter, the corresponding critical dimension for nonlinearities to influence the effects of the initial randomness is $d_u = 2$. This means that, for $d > 2$, the large time fluctuations deriving from the initial state randomness are Gaussian in character. However, for $d < 2$ this is not the case. The behaviour for $d < 2$ may be investigated within an expansion in $\epsilon = 2 - d$. There turns out to be a remarkable universality in the spectrum of large time fluctuations in this case. The equal time order parameter fluctuations all scale like $t^{-1/2}$, with an exponent independent of d. However, they are not normally distributed. The distribution of the local order parameter scaled by $t^{1/2}$ has a universal form which becomes increasingly bimodal in shape as d is decreased below two dimensions. The equal time correlations also attain universal forms in the large time limit. The response function, which gives the response of the order parameter at large t to a change in the initial conditions, has a time decay of the form $t^{-(d+\gamma_1^*)/2}$. We have computed the exponent γ_1^* to second order in ϵ. This universal behaviour is insensitive to small modifications in the forms of both the functional F and the distribution of initial state fluctuations. These modifications give rise to perturbations which are irrelevant in the sense of the renormalization group. The exponents characterizing these corrections are computed for the most important perturbations.

Unlike the case of thermal fluctuations, which destroy the low-temperature ordered phase for sufficiently low dimension d, the initial state randomness, since it ultimately decays away, cannot influence the behaviour of the system at very large times. Thus, with the neglect of thermal fluctuations, there is no lower critical dimension for this problem, and there should be non-trivial behaviour all the way down to $d = 0$. Since the problem is easily soluble in this case, this forms an important check of our ϵ-expansion results. In addition, when generalized to an n-component order parameter, the model turns out to be exactly soluble in the $n \to \infty$ limit for all d. This forms a further check on our calculations.

The model we consider has been used extensively in studying the effects of a quench from a high temperature into the ordered phase [9]. In that case, randomness in the initial state is dissipated by the motion of domain walls. We should stress that in this paper we consider exclusively the case of a quench to a temperature at, or just above, the critical temperature, when the slowness in the dynamics is a result of the critical slowing down of local fluctuations. In principle, it would be necessary to combine both types of analysis, as well as incorporate the effects of thermal noise, in considering a quench to just below the critical temperature.

The layout of this paper is as follows. In the next section we develop the field theory formulation of this problem and the diagrammatic expansion used in its analysis. Then, the model is solved in a self-consistent, Hartree-like approximation, whose results are prototypical of what is expected for the full theory. The next section contains the main results of the paper. We develop the renormalization group program for this model, and compute the renormalization group functions to 2-loop order. We show how the structure of renormalization in the theory leads to results for one exponent which are exact to all orders in ϵ. The following section is devoted to a comparison with exact results obtained in $d = 0$ dimensions. Finally, we summarize our conclusions and make some further remarks concerning generalizations of this work and potential applications to other types of dynamical systems.

2. Field theory formulation

As described in section 1, we are interested in solving the partial differential equation

$$\frac{\partial \phi}{\partial t} = -\Gamma\left(-\nabla^2\phi + r\phi + \lambda\phi^3\right) \tag{3}$$

with the initial condition $\phi(x,t) = \phi(x,0)$, where the $\phi(x,0)$ are random variables drawn from a probability distribution satisfying $\overline{\phi(x,0)} = m$, $\overline{\phi(x,0)\phi(x',0)} = m^2 + \Delta\delta(x-x')$. In principle, the delta function could be replaced by some short-ranged function, and the higher cumulants could also be taken onto account, but, as will be discussed later, these modifications are irrelevant for the universal properties of the critical behaviour.

When $\lambda = 0$, equation (3) is the diffusion or heat equation, with solution

$$\phi(x,t) = \int G_0(x-x',t)\phi(x',0)\,\mathrm{d}^d x \tag{4}$$

where $G_0(x,t) = \int \mathrm{e}^{-\Gamma(q^2+r)t}\,\mathrm{d}^d q/(2\pi)^d$. Since the equation is linear, if $\phi(x,0)$ has a Gaussian distribution, so does $\phi(x,t)$ for $t > 0$. The mean order parameter $\overline{\phi(x,t)}$ does not decay in this approximation at the critical point $r = 0$, and the equal-time correlation function is

$$\begin{aligned}\overline{\phi(x_1,t)\phi(x_2,t)} - \overline{\phi(x,t)}^2 &= \Delta\int G_0(x_1-x',t)G_0(x_2-x',t)\,\mathrm{d}^d x' \\ &= \Delta\int \mathrm{e}^{-2\Gamma q^2 t}\mathrm{e}^{\mathrm{i}q\cdot(x_1-x_2)}\frac{\mathrm{d}^d q}{(2\pi)^d}.\end{aligned} \tag{5}$$

Note, in particular, that in this case the local fluctuations $\overline{\phi(x,t)^2} - m^2$ behave like $\sim t^{-d/2}$.

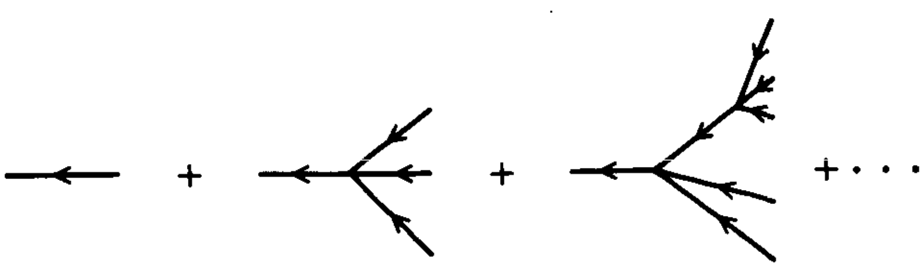

Figure 1. Tree diagrams representing the perturbative solution of equation (3). Each vertex carries a factor of $-\lambda$.

When $\lambda \neq 0$, the solution of equation (3) may be obtained iteratively as a perturbation expansion in λ. Each term in the expansion may be represented by a tree diagram in which each propagator corresponds to the Green function G_0, and each vertex to the interaction $-\lambda$. The result is a set of tree diagrams, the first few of which are shown in figure 1. For example, the second diagram gives a contribution

$$-\lambda\int_0^t \mathrm{d}t'\int \mathrm{d}^d x'\,\mathrm{d}^d x_1\,\mathrm{d}^d x_2\,\mathrm{d}^d x_3 G_0(x-x',t-t')$$
$$\times G_0(x'-x_1,t')G_0(x'-x_2,t')G_0(x'-x_3,t')\phi(x_1,0)\phi(x_2,0)\phi(x_3,0) \tag{6}$$

to $\phi(x,t)$. On averaging over the $\phi(x_i,0)$ with the above distribution, some of the free ends at $t=0$ are sewn together in pairs, with an associated factor of Δ for each pair. The remaining ends each carry a factor of m. Similarly, one may represent other averages, such as the correlation function $\overline{\phi(x,t)\phi(x',t')}$, graphically. Another important quantity is the response function, defined as $\overline{\delta\phi(x,t)/\delta\phi(x',0)}$. This is given by a sum of similar diagrams, with the end at $(x',0)$ left free. The first few diagrams contributing to the response function in the case $m=0$ are shown in figure 2.

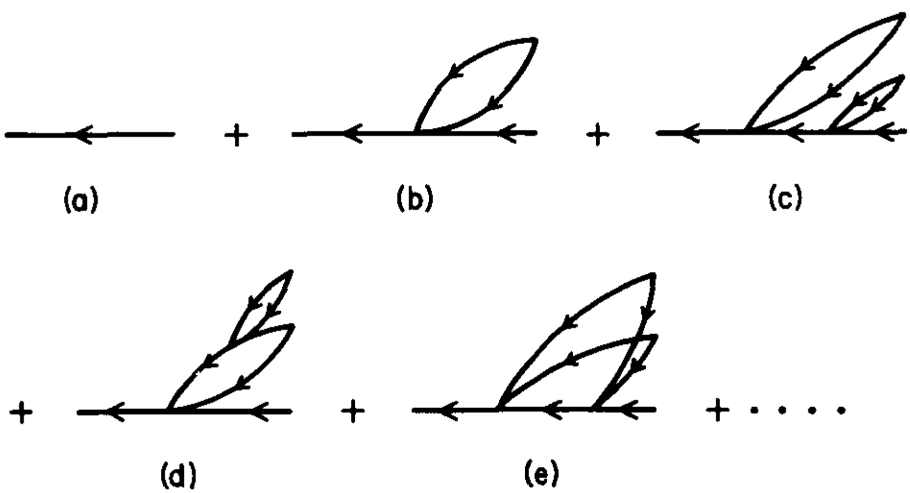

Figure 2. Diagrams, up to 2 loops, contributing to the response function $G_{\phi\tilde{\phi}}$. All closed loops terminate at $t=0$ with a factor of Δ.

Since the problem, after averaging, possesses translational invariance in space, it is convenient to evaluate diagrams in the (q,t) representation, in which everything is Fourier transformed with respect to x. In that case, the propagator is simply $e^{-\Gamma(q^2+r)t}$, and each line carries a 'momentum' q which is integrated over, subject to being conserved at the vertices. The vertices are time-ordered, and integrals are performed over these intermediate times subject to this constraint. In addition, each diagram carries a symmetry factor, corresponding to the number of ways it can be derived following the above procedure of iterating the equation and sewing together the initial ends. While it is possible to give a general rule for this factor, in practice it is straightforward, and more reliable, to derive it from first principles. In some cases it is also convenient to work in the (q,ω) representation, Fourier transforming also with respect to t. Because of the initial condition, however, the problem is not translationally invariant in time, and such a representation is useful only in evaluating correlation functions depending on just one or two time coordinates. For example, when evaluating the response function, we may Fourier transform with respect to t and evaluate diagrams in time-ordered perturbation theory, assigning a propagator $(-i\omega+\sum_i(q_i^2+r))^{-1}$ to each intermediate state consisting of lines carrying momenta q_i.

This set of Feynman rules leads to divergences in some of the integrals, which will be analysed later, and which play a crucial role in the renormalization group analysis. They may be removed by replacing the δ-function correlations in the initial condition by something smoother, but it is more practical to regularize them either by imposing a cut-off $|q| < \Lambda$ in all internal momentum integrals, or by analytic continuation in the spatial dimension d.

These diagrammatic rules may also be derived from an action in a manner similar to that used for studying dynamical critical phenomena [6, 7]. Equation (3) may be imposed with a functional delta function, introducing an auxiliary response field $\tilde{\phi}(x,t)$

$$\int \mathcal{D}\tilde{\phi}\mathcal{D}\phi \exp\left(\int \mathrm{d}^d x\,\mathrm{d}t\tilde{\phi}\left(\dot{\phi}+\Gamma(-\nabla^2\phi+r\phi+\lambda\phi^3)\right)\right). \tag{7}$$

When $\lambda = 0$, $\langle\phi(x,t)\tilde{\phi}(x',0)\rangle$ is just the bare response function $G_0(x-x',t)$. In general, a tree diagram for $\phi(x,t)$ with ends at $(x_1,x_2,\ldots)$ corresponds to $\langle\phi(x,t)\tilde{\phi}(x_1,0)\tilde{\phi}(x_2,0)\ldots\rangle$ evaluated with respect to the measure in equation (7). Thus, the average over the initial conditions $\phi(x,0)$ may be implemented by integrating over $\tilde{\phi}(x,0)$ with a Gaussian weight factor. The result is that the full response function is

$$G_{\phi\tilde{\phi}}(x,t) = \langle\phi(x,t)\tilde{\phi}(0,0)\rangle \tag{8}$$

and the correlation function is

$$G_{\phi\phi}(x_1,t_1;x_2,t_2) = \langle\phi(x_1,t_1)\phi(x_2,t_2)\rangle \tag{9}$$

where the averages $\langle.\rangle$ are computed with respect to the weight e^S, and

$$S = \int \mathrm{d}^d x\,\mathrm{d}t\tilde{\phi}\left(\dot{\phi}+\Gamma(-\nabla^2\phi+r\phi+\lambda\phi^3)\right) - m\int \mathrm{d}^d x\tilde{\phi}(x,0) - \tfrac{1}{2}\Delta\int \mathrm{d}^d x\tilde{\phi}(x,0)^2. \tag{10}$$

Note that this differs from the usual functional integral formalism of critical dynamics [6, 7] only in that the last two terms are localized to the $t = 0$ time slice.

In this form, the problem resembles that of the equilibrium statistical mechanics of a semi-infinite system, where the dimension normal to the boundary is time t. By analogy with such systems [10], we should expect that the critical behaviour of the quantities determining the 'bulk' behaviour for $t > 0$ should not depend on the boundary terms, but that the critical behaviour of correlation functions involving boundary fields may depend on both the bulk and the boundary terms. Since in this case the 'bulk' behaviour corresponds to the fully deterministic equation (3), we expect that the parameters Γ, r and λ which determine this are not renormalized

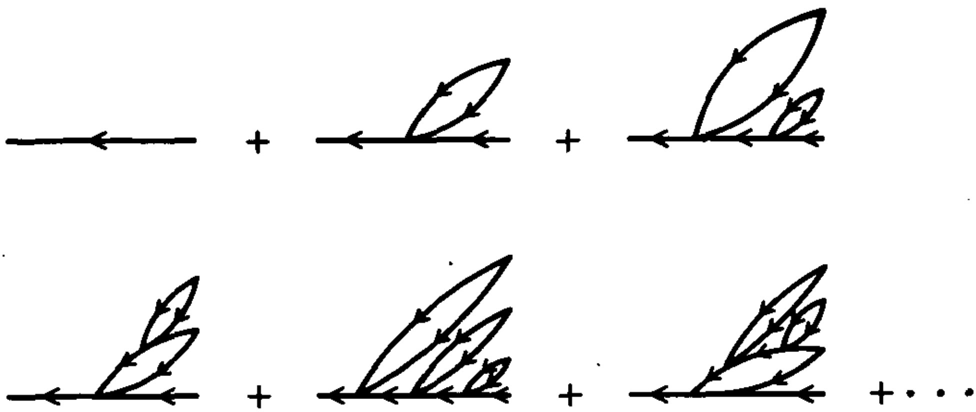

Figure 3. Diagrams surviving in the self-consistent approximation (large n limit).

in any way, but that 'boundary' operators such as $\tilde{\phi}(x,0)$ and boundary coupling constants such as Δ should undergo renormalization, leading to possible non-trivial critical behaviour for response functions. As we shall see, with some refinements, this picture is accurate.

The action functional equation (10) is useful for performing the dimensional analysis which will guide the subsequent renormalization group program. In terms of dimensions of momentum k and frequency ω, we see from equation (10) that

$$[\Gamma] = \omega k^{-2} \qquad [\lambda] = k^2[\phi]^{-2} \qquad [\Delta] = k^{-d}[\phi]^2 \qquad [m] = [\phi] \tag{11}$$

which indicates that the true expansion parameter is $\lambda\Delta$, with dimension k^{2-d}. This suggests that $d = 2$ is the upper critical dimension for this problem, since for $d < 2$ higher order terms in the coupling constant will be accompanied by more and more singular behaviour of the coefficients as $t \to \infty$. In anticipation of the result, suggested above, that Γ and λ are not renormalized, we may rescale the fields so that they are both set to unity. We then have the following list of engineering dimensions:

$$[\phi] = k \qquad [\tilde{\phi}] = k^{d-1} \qquad [\Delta] = k^{2-d} \qquad [m] = k. \tag{12}$$

The last result shows that the symmetry-breaking parameter is relevant, and should be expected to modify the large-time behaviour in a manner similar to that of a surface magnetic field.

3. Self-consistent calculation

Before going into the details of the renormalization group analysis, we first describe a self-consistent approximate solution to the problem. A similar analysis has been made for the case when thermal noise is included [3], and for a quench down to zero temperature [12]. This involves replacing the nonlinear ϕ^3 term in equation (3) by $3\overline{\phi^2}\cdot\phi$, thus rendering the equation linear, albeit now time-dependent, and calculating $\overline{\phi^2(t)}$ self-consistently. When $m = 0$, this approximation is equivalent to summing the set of diagrams shown in figure 3. As will be shown later, this becomes exact in the large n limit, with the replacement $3 \to n + 2$.

The solution of the differential equation has the form

$$\phi(x,t) = \int G_{\mathrm{SC}}(x-x',t)\phi(x',0)\,\mathrm{d}^d x \tag{13}$$

where the Fourier transform of the self-consistent response function satisfies the integral equation

$$G_{\mathrm{SC}}(q,t) = \mathrm{e}^{-(q^2+r)t} - 3\int_0^t \mathrm{e}^{-(q^2+r)(t-t')}\overline{\phi(t)^2}G_{\mathrm{SC}}(q,t')\,\mathrm{d}t'. \tag{14}$$

From the form of this equation it is clear that G_{SC} has the functional form

$$G_{\mathrm{SC}}(q,t) = \mathrm{e}^{-(q^2+r)t}F(t) \tag{15}$$

so that

$$\overline{\phi(t)^2} = F(t)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{e}^{-2(q^2+r)t}\left(m^2(2\pi)^d\delta(q)+\Delta\right). \tag{16}$$

Hence $F(t)$ satisfies the simple integral equation

$$F(t) = 1 - 3\int_0^t \mathrm{e}^{-2rt'}\left(m^2 + \frac{\Delta}{(8\pi)^{d/2}t^{d/2}}\right)F(t')^3\,\mathrm{d}t' \tag{17}$$

the solution of which is of the form

$$F(t)^{-2} = 1 + 6\int^t \mathrm{e}^{-2rt'}\left(m^2 + \frac{\Delta}{(8\pi)^{d/2}t^{d/2}}\right)\mathrm{d}t'. \tag{18}$$

The role of $d = 2$ is apparent in this expression. For $d < 2$ the perturbation expansion is well-behaved in the small-time limit, and we may take the lower limit of the t' integration to be zero. For $d \geqslant 2$ this is no longer true. An analysis of the diagrams in figure 3 shows that they are finite with a momentum cut-off $|k| < \Lambda$ if the time integrations are performed first. Since we have not done this, it is necessary instead to cut off the integral in equation (17) at $t' \sim \Lambda^{-1/2}$. With this in mind, it is possible to analyse the large t behaviour of the solution. For $r > 0$, we see that $F(t) \to$ constant, so that G_{SC} behaves essentially as in the non-interacting case, with exponential decay of the initial state fluctuations. At the critical point $r = 0$, we see that, for $m = 0$ and $d < 2$, $F(t)$ has the form

$$F(t) = \frac{1}{\left(1 + \mathrm{constant}\,\Delta t^{1-d/2}\right)^{1/2}} \sim t^{-\epsilon/4} \tag{19}$$

where $d = 2 - \epsilon$. It is interesting to compute the equal-time correlation function, which, in the same approximation, is given in q-space by $\Delta G_{\mathrm{SC}}(q,t)^2$. In particular, we see that

$$\overline{\phi(x,t)^2} \sim \Delta F(t)^2 \int \mathrm{e}^{-2q^2 t}\frac{\mathrm{d}^d q}{(2\pi)^d} \sim \frac{\epsilon}{12t}. \tag{20}$$

Later we shall show that this hyperuniversal exponent, independent of d for $d < 2$, is a general feature and not limited to this approximation. It is interesting to compare this result with the non-interacting case which gives $t^{-d/2}$. The amplitude in equation (20) will also be shown to be universal, and correct to first order in ϵ.

Next, consider the case when $r = 0$ but $m \neq 0$, so that the initial conditions break the symmetry on average. Then we see that $F(t) \sim 1/(mt^{1/2})$, and the fluctuations are relatively less significant. However, there is an intermediate regime described by a scaling form

$$F(t) \sim t^{-\epsilon/4}\Psi\left(m^2 t^{d/2}/\Delta\right) \tag{21}$$

where $\Psi(u) = (1+\text{constant}\, u)^{-1/2}$. This illustrates that m acts as a relevant variable with crossover exponent $\frac{1}{2}d$.

Another quantity of interest is the response function $\langle\phi(x,t)\tilde{\phi}(x',t_0)\rangle$ for $t_0 > 0$. It satisfies a similar integral equation to that for G_{SC}, and its Fourier transform has the form, at criticality, $\mathrm{e}^{-q^2(t-t_0)}F(t;t_0)$, where now

$$F(t;t_0) = 1 - \frac{\Delta}{(8\pi)^{d/2}}\int_{t_0}^{t}\frac{F(t')^2F(t';t_0)}{t'^{d/2}}\mathrm{d}t'. \tag{22}$$

The solution is

$$F(t;t_0) = \exp -(\Delta/(8\pi)^{d/2})\int_{t_0}^{t} t'^{-d/2}F(t')^2\, dt' \tag{23}$$

and, on substituting the exact form for $F(t')$, we find that

$$F(t;t_0) = \left(\frac{t}{t_0}\right)^{(d-2)/4}. \tag{24}$$

Thus, $F(t;t_0)$ has the same behaviour as $t\to\infty$ at fixed t_0 as does $F(t)$; however, it has a different overall dimension, and its limit as $t_0 \to 0$ is *not* $F(t)$. This is an example of the different renormalization effects for $t = 0$ and $t > 0$, in analogy with surface critical behaviour [10], which will be commented upon later.

Finally, in the low-temperature phase $r < 0$ we see from equation (17) that $F(t)^{-2} \sim \left(m^2 + \mathrm{O}(t^{-d/2})\right)\mathrm{e}^{2|r|t}$ as $t \to \infty$, so that the leading behaviour of the bare term is cancelled, and

$$G_{\mathrm{SC}}(q,t) \sim \begin{cases} \mathrm{e}^{-q^2t} & \text{if } m \neq 0 \\ t^{d/4}\mathrm{e}^{-q^2t} & \text{if } m = 0. \end{cases} \tag{25}$$

This implies that for $m = 0$ the response function $\langle\phi(x,t)\tilde{\phi}(x,0)\rangle$ has a slow $t^{-d/4}$ decay, presumably reflecting the slow diffusion of domain walls. However, the local fluctuations $\overline{\phi(x,t)^2} \sim t^0$, as expected in an ordered phase.

4. Renormalization group analysis

4.1. Analysis of divergences

We begin the renormalization group program by making an analysis of the divergences occurring in perturbation theory [8]. A general correlation function of non-composite fields has the form

$$\langle\phi(x_1,t_1)\phi(x_2,t_2)\ldots\phi(x_l,t_l)\cdot\widetilde{\phi}(x_1',t_1')\widetilde{\phi}(x_2',t_2')\ldots\widetilde{\phi}(x_m',t_m')\rangle \tag{26}$$

calculated with respect to the action equation (10). Without lack of generality, we may assume that all the arguments t_i of the fields ϕ are strictly positive. It is convenient to work in the (q,t) representation, since the bare propagator is then dimensionless, and the power counting of divergences is simpler. We first identify the primitively divergent diagrams, that is those divergent diagrams with no divergent subdiagrams, using dimensional analysis. From equation (12) we see that the Fourier transform of the above correlation function has dimensions

$$k^d\cdot k^{-(l+m)d}\cdot k^l\cdot k^{m(d-1)}=k^{d-(d-1)l-m} \tag{27}$$

where the first factor comes from factoring out an overall momentum-conserving delta function, and the second from the Fourier integrals. If this correlation function is then expanded in powers of Δ, the Feynman integrals contributing to the coefficient of Δ^n will have dimension k^δ, where

$$\delta=k^{d-(d-1)l-m+(d-2)n}. \tag{28}$$

When δ is positive, the corresponding diagram is potentially divergent, with the appropriate power of the cut-off Λ. For $d>2$ we see that, no matter what the values of l and m, arbitrarily high powers of Λ appear, indicating that the theory is not renormalizable, or, equivalently, that Δ is irrelevant for the large t behaviour. For the marginal case $d=2$, $\delta<0$ except for $\langle\phi(x,t)\widetilde{\phi}(x',t')\rangle$ and $G_{\phi\phi}=\langle\phi(x_1,t_1)\phi(x_2,t_2)\rangle$. This implies that these are the only correlation functions with primitive divergences. This dimensional argument assumes that the divergence is arising from all the internal momenta of a given diagram becoming simultaneously large. Divergences may also arise when a subset become large, but this is, by definition, a divergence in a subdiagram and so is not primitive. We shall assume, in accordance with conventional field theories, that renormalization of the primitively divergent diagrams is sufficient to render all diagrams finite [8].

In fact, $\langle\phi(x,t)\widetilde{\phi}(x',t')\rangle$ is not primitively divergent for $t'>0$. This is because at least one of the integrations over the time of an internal vertex will have a lower bound of t', and the associated propagator entering this vertex, carrying momentum k, will behave at large k like $e^{-k^2t'}$, and so will be exponentially damped. Thus, if the diagram is divergent, this divergence must come from a subdiagram. Since this would also apply to any correlation function of the form equation (26) in which all the arguments t_i and t_j' are positive, it follows that neither $\phi(x,t)$ nor $\widetilde{\phi}(x',t')$ need any multiplicative renormalization for $t,t'>0$, and that, as stated earlier, neither will other parameters characterizing the $t>0$ theory, such as λ and Γ. Only a coupling constant renormalization is needed, in principle, to render finite the correlation functions with all time arguments positive. However, $\tilde{\phi}(x',0)$ may need multiplicative wavefunction renormalization in its correlation functions, in addition to coupling constant renormalization. As stated earlier, we never need to consider correlation functions involving $\phi(x,0)$.

We conclude that renormalization in this theory should be very simple, and that it involves a multiplicative renormalization of $\tilde{\phi}(x',0)$ and a renormalization of Δ. Although these arguments are somewhat formal, and certainly not rigorous, we shall see that, at least to 2-loop order, they are correct. The fact that $\tilde{\phi}(x',t')$ renormalizes differently for $t'>0$ and $t'=0$ is familiar from the analogous case of surface critical behaviour [10]. In our case, the 'bulk' theory corresponds to a deterministic equation, for which there can be no renormalization effects.

4.2. *Two-loop calculation*

Guided by the results of the last section, we now consider the diagrams contributing to $G_{\phi\tilde{\phi}}$ and $G_{\phi\phi}$, which are the only ones containing primitive divergences. We shall consider the critical theory, and work in $d=2-\epsilon$ dimensions, which will serve to regulate all divergences.

First, consider $G_{\phi\tilde{\phi}}(q,\omega)=\int \mathrm{d}^d x\, \mathrm{d}t \mathrm{e}^{\mathrm{i}q\cdot x}\mathrm{e}^{\mathrm{i}\omega t}\langle\phi(x,t)\tilde{\phi}(0,0)\rangle$. The diagrams contributing to this up to two loops are shown in figure 2. The bare term is $(-\mathrm{i}\omega+q^2)^{-1}$. The 1-loop contribution is that of diagram (b). It is

$$I_b=-3\Delta\frac{1}{-\mathrm{i}\omega+q^2}\int\frac{\mathbf{d}k}{-\mathrm{i}\omega+q^2+2k^2} \tag{29}$$

where $\mathbf{d}k$ is shorthand for $\mathrm{d}^d k/(2\pi)^d$. The factor of 3 arises from the three ways of pairing the end vertices of the associated tree diagram. The integral is readily evaluated by standard methods to give

$$I_b=3\Delta\left(\frac{S_2}{2\epsilon}+\mathrm{O}(\epsilon)\right)\frac{1}{-\mathrm{i}\omega+q^2}\left(\frac{-\mathrm{i}\omega+q^2}{2}\right)^{-\epsilon/2} \tag{30}$$

where $S_2=\frac{1}{2}\pi$. This factor is ubiquitous and, henceforth, will be absorbed into the definition of the coupling constant. Similarly we find that

$$\begin{aligned}I_d=2I_c&=9(2\pi\Delta)^2\frac{1}{-\mathrm{i}\omega+q^2}\int\frac{\mathbf{d}k_1\mathbf{d}k_2}{(-\mathrm{i}\omega+q^2+2k_1^2)(-\mathrm{i}\omega+q^2+2k_1^2+2k_2^2)}\\&=9\Delta^2\left(\frac{1}{4\epsilon^2}+\mathrm{O}(\epsilon^0)\right)\frac{1}{-\mathrm{i}\omega+q^2}\left(\frac{-\mathrm{i}\omega+q^2}{2}\right)^{-\epsilon}.\end{aligned} \tag{31}$$

The only non-trivial 2-loop integral is that of figure 2(*e*). This is evaluated in appendix B, to give

$$I_e=\frac{9}{4}\ln 3\,\Delta^2\left(\frac{1}{2\epsilon}+\mathrm{O}(1)\right)\frac{1}{-\mathrm{i}\omega+q^2}\left(\left(\frac{-\mathrm{i}\omega+q^2}{2}\right)^{-\epsilon}+\mathrm{O}(\epsilon)\right). \tag{32}$$

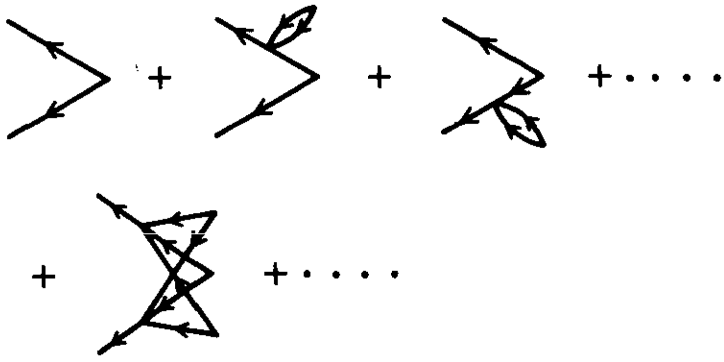

Figure 4. Diagrams, up to two loops, contributing to the correlation function $G_{\phi\phi}$. Only (f) is a new Feynman integral.

Next, consider

$$G_{\phi\phi}(q,\omega_1,\omega_2) = \int \mathrm{d}^d x\,\mathrm{d}t_1\,\mathrm{d}t_2 \mathrm{e}^{\mathrm{i}q\cdot x}\mathrm{e}^{\mathrm{i}\omega_1 t_1}\mathrm{e}^{\mathrm{i}\omega_2 t_2}\langle\phi(x,t_1)\phi(0,t_2)\rangle.$$

The bare term is $2\pi\Delta(-\mathrm{i}\omega_1+q^2)^{-1}(-\mathrm{i}\omega_2+q^2)^{-1}$, and the corrections up to two loops are shown in figure 4. The calculation of the first few diagrams is facilitated by observing that

$$G_{\phi\phi}(q,\omega_1,\omega_2) = 2\pi\Delta G_{\phi\tilde{\phi}}(q,\omega_1)G_{\phi\tilde{\phi}}(-q,\omega_2) + \mathrm{O}(\Delta^3) \tag{33}$$

with the only $\mathrm{O}(\Delta^3)$ correction coming from diagram (f), which is evaluated in appendix B to give

$$I_f = \Delta^2\kappa^{-2\epsilon}\left(\frac{1}{2\epsilon}+\mathrm{O}(1)\right)\frac{1}{(-\mathrm{i}\omega_1+q^2)(-\mathrm{i}\omega_2+q^2)}. \tag{34}$$

Since $G_{\phi\tilde{\phi}}$ and $G_{\phi\phi}$ have only logarithmic divergences at $d=2$, to renormalize them we need do so only at one point. It is convenient to choose this to be $\omega_1=\omega_2=0$ and $q=\kappa$, where k is arbitrary. We may then summarize the results of the 2-loop calculation as

$$G_{\phi\tilde{\phi}} = \kappa^{-2}\left(1-\frac{3}{2}\Delta\frac{\kappa^{-\epsilon}}{\epsilon}+\frac{27}{8}\Delta^2\frac{\kappa^{-2\epsilon}}{\epsilon^2}+\frac{9}{4}\ln 3\,\Delta^2\frac{\kappa^{-2\epsilon}}{2\epsilon}+\mathrm{O}(\Delta^3)\right) \tag{35}$$

and

$$G_{\phi\phi} = 2\pi\Delta\kappa^{-4}\left(1-3\Delta\frac{\kappa^{-\epsilon}}{\epsilon}+9\Delta^2\frac{\kappa^{-2\epsilon}}{\epsilon^2}+\left(\frac{9}{2}\ln 3+1\right)\Delta^2\frac{\kappa^{-\epsilon}}{2\epsilon}+\mathrm{O}(\Delta^3)\right) \tag{36}$$

4.3. Renormalization and ϵ expansion

As discussed above, we expect that the whole theory, including $G_{\phi\tilde{\phi}}$ and $G_{\phi\phi}$, may be rendered finite by suitably renormalizing $\tilde{\phi}(t = 0)$ and Δ. Specifically, define $\tilde{\phi}_{\mathrm{R}} = Z_1^{-1}\tilde{\phi}$ by the condition that $\langle\phi\tilde{\phi}_{\mathrm{R}}\rangle = \kappa^{-2}$ at the normalization point $\omega = 0$, $q = \kappa$, and $\Delta_{\mathrm{R}} = Z_2\Delta$ by the condition that $\langle\phi\phi\rangle = 2\pi\Delta_{\mathrm{R}}\kappa^{-4}$ at $\omega_1 = \omega_2 = 0$, $q = \kappa$. Then Z_1 and Z_2 are given by the expressions in parentheses in equations (35) and (36) respectively.

It is also useful to define the dimensionless renormalized coupling by $g_{\mathrm{R}} = \Delta_{\mathrm{R}}\kappa^{-\epsilon}$. We shall also need the renormalization group functions

$$\gamma_1 = \kappa \left.\frac{\partial \ln Z_1}{\partial\kappa}\right|_{\Delta} \qquad \gamma_2 = \kappa \left.\frac{\partial \ln Z_2}{\partial\kappa}\right|_{\Delta} \tag{37}$$

and

$$\beta(g_{\mathrm{R}}) = \kappa \left.\frac{\partial g_{\mathrm{R}}}{\partial\kappa}\right|_{\Delta} . \tag{38}$$

Note that

$$\beta(g_{\mathrm{R}}) = -\epsilon g_{\mathrm{R}} + \gamma_2(g_{\mathrm{R}}). \tag{39}$$

Explicitly we have

$$\begin{aligned}\gamma_1 &= \frac{3}{2}\Delta\kappa^{-\epsilon} - \frac{9}{2}\Delta^2\frac{\kappa^{-2\epsilon}}{\epsilon} - \frac{9}{4}\ln 3\Delta^2\kappa^{-2\epsilon} + \cdots \\ &= \tfrac{3}{2}g_{\mathrm{R}} - \tfrac{9}{4}\ln 3 g_{\mathrm{R}}^2 + \cdots\end{aligned} \tag{40}$$

and

$$\begin{aligned}\gamma_2 &= 3\Delta\kappa^{-\epsilon} - 9\Delta^2\frac{\kappa^{-2\epsilon}}{\epsilon} - (\tfrac{9}{2}\ln 3 + 1)\Delta^2\kappa^{-2\epsilon} + \cdots \\ &= 3g_{\mathrm{R}} - (\tfrac{9}{2}\ln 3 + 1)g_{\mathrm{R}}^2 + \cdots.\end{aligned} \tag{41}$$

Note that the fact that the renormalization group functions are finite as $\epsilon \to 0$ when expressed in terms of the renormalized coupling constant is an important and non-trivial check of our assertion that coupling constant renormalization is sufficient to render $G_{\phi\phi}$ finite.

We now write down the Callan–Symanzik equations for $G_{\phi\tilde{\phi}}$ and $G_{\phi\phi}$ by observing that the bare quantities do not depend on κ. Thus

$$\kappa\frac{\partial}{\partial\kappa}\left(Z_1 G^{\mathrm{R}}_{\phi\tilde{\phi}}\right) = \kappa\frac{\partial}{\partial\kappa}\left(Z_2\Delta_{\mathrm{R}}^{-1}G^{\mathrm{R}}_{\phi\phi}\right) = 0 \tag{42}$$

which leads to

$$\left(\kappa\frac{\partial}{\partial\kappa} + \gamma_1(g_{\mathrm{R}}) + \beta(g_{\mathrm{R}})\frac{\partial}{\partial g_{\mathrm{R}}}\right) G^{\mathrm{R}}_{\phi\tilde{\phi}} = 0 \tag{43}$$

$$\left(\kappa\frac{\partial}{\partial\kappa} + \gamma_2(g_{\mathrm{R}}) + \beta(g_{\mathrm{R}})\frac{\partial}{\partial g_{\mathrm{R}}}\right) (\Delta_{\mathrm{R}}^{-1}G^{\mathrm{R}}_{\phi\phi}) = 0. \tag{44}$$

The standard solution of these equations shows that the $\kappa \to 0$ limit of the correlation functions is controlled by a value g_R^* of g_R at an infrared stable zero of $\beta(g_R)$, that is one where $\beta'(g_R^*) > 0$. From equations (39) and (41) we see that such a zero, of order ϵ, exists for $\epsilon > 0$. At such a fixed point,

$$G^R_{\phi\tilde{\phi}}(q,\omega,\kappa) \sim \kappa^{-\gamma_1^*} \tag{45}$$

$$\Delta_R^{-1} G^R_{\phi\phi}(q,\omega_1,\omega_2,\kappa) \sim \kappa^{-\gamma_2^*} \tag{46}$$

at fixed (q,ω), where $\gamma_i^* = \gamma_i(g_R^*)$. Since dimensionally $[G_{\phi\tilde{\phi}}(q,\omega,\kappa)] = k^{-2}$ and $[\Delta^{-1}G_{\phi\phi}(q,\omega_1,\omega_2,\kappa)] = k^{-4}$, we see that, in the (q,t) representation,

$$G_{\phi\tilde{\phi}}(q,t) \sim t^{-\gamma_1^*/2} f_1(q^2 t) \quad (47) \qquad G_{\phi\phi}(q,t,t) \sim \Delta t^{-\gamma_2^*/2} f_2(q^2 t) \quad (48)$$

where f_1 and f_2 are scaling functions. Solving equations (39–41) we then find explicitly that

$$\gamma_1^* = \frac{\epsilon}{2} + \frac{\epsilon^2}{18} + O(\epsilon^3) \tag{49}$$

$$\gamma_2^* = \epsilon. \tag{50}$$

It is interesting to observe that the contribution of figure 2(*e*) cancels in the final result for γ_1^*. Note also that, because of equation (39), the result for γ_2^* is valid to all orders in ϵ. This has the important consequence that if we compute the exponent governing the time dependence of the local fluctuations

$$\overline{\phi(x,t)^2} = \int G_{\phi\phi}(q,t,t)\,\mathrm{d}^d q \sim t^{-1} \tag{51}$$

the same d-independent exponent as was found in the self-consistent approximation.

This result may be traced back to the fact that ϕ is not renormalized, so its correlation functions scale with its engineering dimension $[\phi] = k$. This implies further that a general equal-time correlation function has the scaling form

$$\overline{\phi(x_1,t)\phi(x_2,t)\ldots\phi(x_n,t)} \sim t^{-n/2} F_n(|x_i - x_j| t^{-1/2}) \tag{52}$$

and that the random variables $t^{1/2}\phi(x,t)$ have a joint probability distribution which is independent of t, if the x-coordinates are scaled appropriately.

It is also interesting to consider the scaling behaviour of the response function $\langle\phi(x,t)\tilde{\phi}(0,t')\rangle$ for $t' > 0$. Denote its Fourier transform with respect to x by $G_{\phi\tilde{\phi}}(q,t,t')$. Since $\tilde{\phi}(x',t')$ is not renormalized for $t' > 0$, this will satisfy a Callan–Symanzik equation similar to equation (43), but with no term involving $\gamma_1(g_R)$. Thus, there are no anomalous dimensions involved, and therefore, at fixed t and t', $G_{\phi\tilde{\phi}} \sim \kappa^0$. However, from dimensional analysis we may now conclude only that

$$G_{\phi\tilde{\phi}}(q,t,t') = \Phi(q^2 t, q^2 t') \tag{53}$$

where Φ is a scaling function yet to be determined. This does not yield information on the behaviour as $t \to \infty$ at fixed t'. In analogy with surface critical behaviour

[10], we expect that this may be investigated by showing the existence of a short-time expansion in which renormalized fields for $t' > 0$ may be expressed in terms of renormalized zero-time fields. In this case, the relevant expansion will have the form

$$\tilde{\phi}(x',t') = At'^{\gamma_1^*/2}\tilde{\phi}_{\mathrm{R}}(x',0) + \cdots \tag{54}$$

where A is some amplitude. This would imply that, for $t'/t \ll 1$,

$$G_{\phi\tilde{\phi}}(q,t,t') \sim (t/t')^{-\gamma_1^*/2} f(q^2 t) \tag{55}$$

as was found in the self-consistent approximation.

The renormalization group analysis readily generalizes to the case when $m \neq 0$. In fact, no new renormalization constants are needed to deal with this case. This is because m couples to $\tilde{\phi}(t=0)$ in the effective action, and therefore the renormalization group eigenvalue of m at the $m = 0$ fixed point is simply related to γ_1^*. The simplest way to derive this relation is to observe that, for $m \neq 0$, we expect $\overline{\phi(x,t)}$ to satisfy the scaling law (see equation (21))

$$\overline{\phi(x,t)} \sim t^{-1/2}\Psi(mt^{\alpha}, x^2/t) \tag{56}$$

where Ψ is scaling function and α is an exponent to be determined. This scaling form may in fact be derived in the standard way [8] by expanding the left-hand side in powers of m, each coefficient being proportional to a correlation function with insertions of $\tilde{\phi}(x,0)$, which satisfies a Callan–Symanzik equation. However, to find α we may simply observe that the first derivative with respect to m, evaluated at $m = 0$, is nothing but the $q = 0$ limit of $G_{\phi\tilde{\phi}}$. Thus, from equation (47), we have the result

$$\alpha = (1 - \gamma_1^*)/2. \tag{57}$$

The scaling function Ψ may also be computed within the ϵ expansion. We expect, for large values of the argument mt^{α}, that the effects of the random initial conditions are unimportant, and that the order parameter relaxes with a $t^{-1/2}$ power law.

4.4. Universal amplitudes

We have shown that the moments of $\phi(x,t)$ scale in universal manner. This leads us to define the amplitudes A_{2n} of the cumulants by

$$\begin{aligned} &\overline{\phi(x,t)^2} \sim A_2/t \\ &\overline{\phi(x,t)^4} - 3\overline{\phi(x,t)^2}^2 \sim A_4/t^2 \end{aligned} \tag{58}$$

and so on†. The moments satisfy the renormalization group equations

$$\left(t\frac{\partial}{\partial t} - \frac{1}{2}\beta(g_{\mathrm{R}})\frac{\partial}{\partial g_{\mathrm{R}}} - n\right)\overline{\phi(x,t)^{2n}} = 0 \tag{59}$$

† These amplitudes are stricitly universal only if $\lambda = 1$. In general, they contain a factor $\lambda^{-n/2}$.

so that, in order to evaluate A_{2n}, we may calculate $\overline{\phi(x,t)^{2n}}$ in renormalized perturbation theory at the normalization scale $t = \kappa^{-2}$, and set g_{R} to its fixed point value.

To lowest order, we have simply

$$t\overline{\phi(x,t)^2}\Big|_{t=\kappa^{-2}} = 2\pi\Delta\kappa^{-2}\int \mathrm{e}^{-2k^2\kappa^{-2}}\mathrm{d}k \sim \frac{\Delta\kappa^{-\epsilon}}{4} \sim \frac{g_{\mathrm{R}}}{4} + \mathrm{O}(g_{\mathrm{R}}^2) \tag{60}$$

so we see that

$$A_2 = \frac{\epsilon}{12} + \mathrm{O}(\epsilon^2). \tag{61}$$

In fact, if we had defined the coupling constant renormalization by $\Delta_{\mathrm{R}} = 4\kappa^{-2}\overline{\phi(x,\kappa^{-2})^2}$, we would have found that $A_2 = g_{\mathrm{R}}^*/4$ to all orders. However, the differences between this scheme and the one used in the previous section show up only at $\mathrm{O}(\epsilon^3)$ in the amplitude A_2. To see this, observe that the ϵ dependence of the perturbation expansion has the form

$$A_2 = \frac{1}{4}\left[\Delta\kappa^{-\epsilon} + a_1(\epsilon)\Delta^2\frac{\kappa^{-2\epsilon}}{\epsilon} + a_2(\epsilon)\Delta^3\frac{\kappa^{-3}}{\epsilon^2} + a_3(\epsilon)\Delta^3\frac{\kappa^{-2\epsilon}}{\epsilon} + \mathrm{O}(\Delta^4)\right] \tag{62}$$

where $a_1 = -3 + \mathrm{O}(\epsilon^2)$ and $a_2 = 9 + \mathrm{O}(\epsilon^2)$. On renormalizing, the pole terms are completely removed, so that

$$A_2 = \frac{g_{\mathrm{R}}}{8\pi} + \mathrm{O}(\epsilon g_{\mathrm{R}}^2) + \mathrm{O}(g_{\mathrm{R}}^3) \tag{63}$$

evaluated at the fixed point. Thus

$$A_2 = \frac{g_{\mathrm{R}}^*}{8\pi} + \mathrm{O}(\epsilon^3) = \frac{\epsilon}{12} + \left(\frac{9}{2}\ln 3 + 1\right)\frac{\epsilon^2}{108} + \mathrm{O}(\epsilon^3). \tag{64}$$

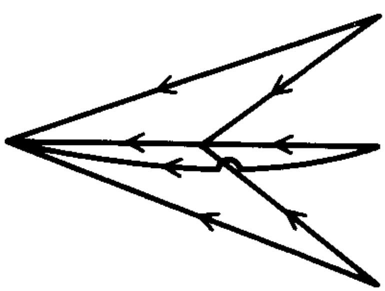

Figure 5. Lowest order diagram contributing to the amplitude A_4 of the fourth cumulant.

Next, consider the amplitude A_4 of the fourth cumulant. In the Gaussian approximation, this vanishes, and it will continue to vanish up to an order where the interactions begin to correlate all four propagators emerging from the ϕ^4 vertex. The lowest order such diagram is shown in figure 5. Note that it is $\mathrm{O}(\Delta^3)$ and therefore negative. The diagram is evaluated in appendix B, to give

$$A_4 = -\frac{3 + 4\ln 2}{6}\epsilon^3 + \mathrm{O}(\epsilon^4). \tag{65}$$

If the negative sign for A_4 persists in higher orders, it implies that the distribution of $\phi(x,t)$ has a tendency to become bimodal, even though the initial distribution function is normal. We shall see that this tendency becomes extreme for $d = 0$.

4.5. Rigorous bounds

The result $\gamma_2^* = \epsilon$, valid to all orders in ϵ, actually saturates a rigorous bound on this exponent. This may be derived as follows.

The equal time correlation function $G_{\phi\phi}$ has the scaling form

$$G_{\phi\phi}(x,t) \sim t^{-(d+\gamma_2^*)/2}\tilde{f}_2(x^2/t) \tag{66}$$

so that $\overline{\phi(x,t)^2} \sim t^{-(d+\gamma_2^*)/2}$. Multiplying the basic equation (3) by ϕ, taking the expectation value, and integrating by parts, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\phi^2} = \nabla\overline{(\phi\nabla\phi)} - \overline{(\nabla\phi)^2} - \overline{(\phi^2)^2}. \tag{67}$$

The first term on the right-hand side vanishes by symmetry, and the last term is bounded by $\overline{\phi^2}^2$. Thus

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\phi^2} \leqslant -\overline{\phi^2}^2. \tag{68}$$

Substituting in the scaling form equation (48), this implies that, for large t,

$$\frac{(d+\gamma_2^*)\tilde{f}_2(0)}{t^{(d+\gamma_2^*)/2+1}} \geqslant \frac{\tilde{f}_2(0)^2}{t^{d+\gamma_2^*}} \tag{69}$$

so that

$$\gamma_2^* \geqslant \epsilon. \tag{70}$$

Our renormalization group analysis indicates that for $d > 2$, $\gamma_2^* = 0$, so that this inequality is trivially satisfied. However, for $d \leqslant 2$, the results of the previous section indicate that the inequality is saturated. In that case, the amplitude $\tilde{f}_2(0) = A_2$, and we have the further bound

$$A_2 \leqslant \tfrac{1}{2}. \tag{71}$$

For ϵ small, this is not a stringent bound, but, as will be shown in the next section, it is saturated for $d = 0$.

4.6. Irrelevant operators

In formulating the problem so far, we have neglected two types of terms which may appear in the effective action equation (10). The first type corresponds to higher order powers of ϕ which may have been omitted from the original equation (3). It is easy to see that they are strongly irrelevant for the critical behaviour. For example, consider a term $\lambda_6\phi^5$ on the right-hand side of equation (3). This corresponds to a term $\lambda_6\tilde{\phi}\phi^5$ in the action equation (10). Power counting shows that $[\lambda_6] = [\phi]^{-2} = k^{-2}$, and since ϕ acquires no anomalous dimension, we conclude that the renormalization group eigenvalue of λ_6 is -2 exactly. Thus, such interactions remain irrelevant for all $d < 2$. Similar considerations hold for terms in equation (3) with higher derivatives.

Of course, this conclusion holds only if $\lambda > 0$. When it vanishes, the scaling has to be performed in a different manner.

More interesting are the effects of modifying the initial distribution. This can be in the form of non-zero higher cumulants, or of non-trivial q-dependence in the second cumulant Δ. The most relevant terms of these types in equation (10) are of the form $\Delta_4 \int \widetilde{\phi}(x,0)^4 \mathrm{d}^d x$ and $\widetilde{\Delta}_2 \int (\nabla\widetilde{\phi})^2 \mathrm{d}^d x$. Dimensionally $[\Delta_4] = k^{4-3d}$ and $[\widetilde{\Delta}_2] = k^{-d}$. Thus Δ_4 would appear to become relevant at the Gaussian fixed point for $d < \frac{4}{3}$. Both have eigenvalue -2 when $d = 2$.

First, consider the renormalization of Δ_4. This may be defined through the value of $\langle \prod_{i=1}^4 \phi(q_i, \omega_i) \widetilde{\phi}^4 \rangle$ at the normalization point $\omega_i = 0$, $q_i = \kappa$. To $\mathrm{O}(\Delta^2)$, the integrals are the same as for the renormalization of Δ, and only the combinatorial factors differ. We find that $\Delta_4^{\mathrm{R}} = Z_4 \Delta_4$, where

$$Z_4 = 1 - 6\Delta \frac{\kappa^{-\epsilon}}{\epsilon} + \mathrm{O}(\Delta^2 \epsilon^{-2}) + (9 \ln 3 + 2)\Delta^2 \frac{\kappa^{-2\epsilon}}{2\epsilon} + \cdots. \tag{72}$$

Thus, defining $\beta_4 = \kappa(\partial/\partial\kappa) g_4^{\mathrm{R}}$ where $g_4^{\mathrm{R}} = \Delta_4^{\mathrm{R}} \kappa^{3d-4}$, we find

$$\beta_4 = (2 - 3\epsilon) g_4^{\mathrm{R}} + \left(6 g_{\mathrm{R}} - (9 \ln 3 + 2) g_{\mathrm{R}}^2 + \cdots\right) g_4^{\mathrm{R}} + \cdots \tag{73}$$

so that the renormalization group eigenvalue $-\partial\beta_4/g_4^{\mathrm{R}}$ at the non-trivial fixed point $g_{\mathrm{R}} = g_{\mathrm{R}}^*$, $g_4^{\mathrm{R}} = 0$ is

$$y_4 = -2 + \epsilon + \mathrm{O}(\epsilon^3). \tag{74}$$

We see that the effect of the interactions is to make Δ_4 less relevant. In fact, if we are able to ignore the $\mathrm{O}(\epsilon^3)$ corrections, the normal distribution characterized simply by the second moment appears to be adequate to describe the universal properties all the way down to $d = 0$. This will be confirmed later.

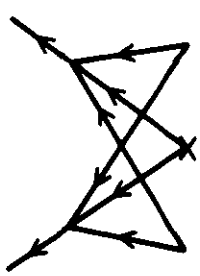

Figure 6. Lowest order dressing of the irrelevant operator $(\nabla\widetilde{\phi})^2$.

Next, consider the renormalization of $\widetilde{\Delta}_2$. The renormalized coupling is defined in terms of $(\partial/\partial q^2)\langle \phi(q, \omega_1)\phi(-q, \omega_2)(\nabla\widetilde{\phi})^2 \rangle$ at the normalization point $\omega_i = 0$, $q^2 = \kappa^2$. To second order in Δ^2, there is one new diagram, shown in figure 6, and evaluated in appendix B. Proceeding as before, we then find that $\widetilde{\Delta}_2^{\mathrm{R}} = \widetilde{Z}_2 \widetilde{\Delta}_2$, where

$$\widetilde{Z}_2 = 1 - 3\Delta \frac{\kappa^{-\epsilon}}{\epsilon} + 9\Delta^2 \frac{\kappa^{-2\epsilon}}{\epsilon^2} + \left(\frac{9}{2} \ln 3 - \frac{1}{27}\right) \Delta^2 \frac{\kappa^{-2\epsilon}}{2\epsilon} + \cdots \tag{75}$$

so that, in analogy with equation (73), we may derive the renormalization group function

$$\tilde{\beta}_2 = (2-\epsilon)\tilde{g}_2^{\mathrm{R}} + \left(3g_{\mathrm{R}} - (\tfrac{9}{2}\ln 3 - \tfrac{1}{27})g_{\mathrm{R}}^2 + \cdots\right)\tilde{g}_2^{\mathrm{R}} + \cdots \tag{76}$$

giving the renormalization group eigenvalue

$$\tilde{y}_2 = -2 - \frac{\epsilon^2}{243} + \mathrm{O}(\epsilon^3). \tag{77}$$

This indicates that the effect of a non-zero, but finite, correlation length in the initial state is highly irrelevant.

4.7. Generalization to n components

So far, we have considered the case of a single component Ising-like order parameter. It is straightforward to generalize the above analysis to a situation with $O(n)$ symmetry, where the order parameter ϕ_i has n components. The basic differential equation is now, in dimensionless units,

$$\dot{\phi}_i = \nabla^2\phi_i - \phi^2\phi_i \tag{78}$$

where $\phi^2 = \sum_j \phi_j^2$. The random initial condition satisfies $\overline{\phi_i(x,0)\phi_j(x',0)} = \Delta\delta_{ij}\delta(x-x')$. In the response field formalism, the vertex $\tilde{\phi}_j\phi_j\phi_i^2$ may be represented as in figure 7. In the resultant diagrams, the indices i are conserved along the full lines, and there is a factor of n for each closed loop. The case $n \to \infty$ is interesting. If the coupling constant is rescaled by $\Delta \to \Delta/n$, the diagrams which survive in this limit are precisely those included in the self-consistent approximation. This is in close analogy with the situation in standard critical behaviour [8].

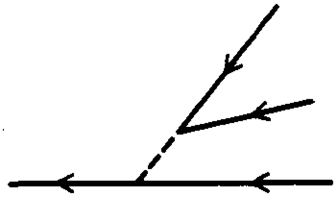

Figure 7. Representation of the interaction vertex in the $O(n)$ generalization.

For finite n, it is straightforward to compute the relevant combinatorial factors up to two loops. The relevant replacements relative to the $n = 1$ case are

$$\begin{aligned} I_b &\to \tfrac{1}{3}(n+2)I_b \\ I_c &\to \tfrac{1}{9}(n+2)^2 I_c \\ I_d &\to \tfrac{1}{9}(n+2)^2 I_d \\ I_e &\to \tfrac{1}{3}(n+2)I_e \\ I_j &\to \tfrac{1}{3}(n+2)I_j. \end{aligned} \tag{79}$$

Thus the renormalization group functions become

$$\gamma_1 = \tfrac{1}{2}(n+2)g_{\mathrm{R}} - \tfrac{3}{4}(n+2)\ln 3 g_{\mathrm{R}}^2 + \cdots \tag{80}$$

$$\gamma_2 = (n+2)g_{\mathrm{R}} - \tfrac{1}{3}(n+2)(\tfrac{9}{2}\ln 3 + 1)g_{\mathrm{R}}^2 + \cdots \tag{81}$$

and the fixed point values are

$$\gamma_1^* = \frac{\epsilon}{2} + \frac{\epsilon^2}{6(n+2)} + \mathrm{O}(\epsilon^3) \tag{82}$$

with $\gamma_2^* = \epsilon$ to all orders, as before. Thus, we see that rescaling $g_{\mathrm{R}} \to g_{\mathrm{R}}/n$ makes the 1-loop results exact as $n \to \infty$, and once again, we recover the results of the self-consistent theory in that limit. The fact that, to the order given above, the renormalization group functions are proportional to $(n+2)$ results from the fact that they are linear in n and must vanish when $n = -2$; for in that case one may argue, as for the equilibrium case, that the two-point functions $\langle\phi\tilde{\phi}\rangle$ and $\langle\phi\phi\rangle$ have no loop corrections [11].

5. Comparison with zero dimensions

We have shown that the theory exhibits a non-trivial fixed point for $d < 2$, in an expansion in $\epsilon = 2 - d$, and that, in particular, the exponent $\gamma_2^* = \epsilon$ to all orders. It is a non-trivial test to check whether this is valid until $\epsilon = 2$, that is $d = 0$. In that case equation (78) becomes a system of n ordinary differential equations, whose solution is elementary:

$$\phi_i(t) = \frac{\phi_i(0)}{(1 + 2t\phi(0)^2)^{1/2}} \tag{83}$$

where $\phi(0)^2 = \sum_j \phi_j(0)^2$. Thus we see that, as $t \to \infty$,

$$\phi_i(t) \sim \frac{1}{(2t)^{1/2}} \frac{\phi_i(0)}{|\phi(0)|} \tag{84}$$

and that, for a random distribution of initial conditions,

$$\overline{\phi_i(t)\phi_j(t)} \sim \frac{\delta_{ij}}{2nt} \tag{85}$$

in agreement, for large n, with the result equation (61), established within the ϵ expansion. It is interesting to note that the amplitude is independent of the form of the initial distribution, indicating the universality of the amplitude A_2. The fourth moment is also easily obtained as

$$\overline{\phi_i(t)^4} \sim \frac{1}{(2t)^2} \frac{\int_0^\pi \cos^4\theta \sin^{n-2}\theta \mathrm{d}\theta}{\int_0^\pi \sin^{n-2}\theta \mathrm{d}\theta} \sim \frac{3}{n(n+2)} \frac{1}{(2t)^2} \tag{86}$$

so that the ratio of the fourth cumulant to the square of the second is

$$\frac{A_4}{A_2^2} = -\frac{6}{n+2}. \tag{87}$$

This result illustrates the interpolation between Gaussian fluctuations ($A_4 = 0$), for large n, and an extreme bimodal distribution for $n = 1$.

Similarly, we may calculate the response function

$$\frac{\partial \phi_i(t)}{\partial \phi_j(0)} = \frac{(1 + 2t\phi(0)^2)\delta_{ij} - 2t\phi_i(0)\phi_j(0)}{(1 + 2t\phi(0)^2)^{3/2}} \tag{88}$$

which, on averaging, gives

$$G_{\phi\tilde{\phi}}(t) \sim t^{-1/2} \tag{89}$$

corresponding to an exponent $\gamma_1^* = 1$, which should be compared with the ϵ-expansion result equation (49). The fact that the $O(\epsilon)$ result is exact for $\epsilon = 2$ may be traced to the property that it is independent of n, and is exact in the limit $n \to \infty$.

Although the model is soluble for $d = 0$, it is interesting to carry through the renormalization group analysis of the previous sections. We choose $n = 1$ for simplicity. Define the renormalized coupling by

$$\Delta_R = \overline{\phi(t)^2}\Big|_{t=\frac{1}{2}\kappa^{-2}} \tag{90}$$

(the factor of $\frac{1}{2}$ is for convenience), and the dimensionless renormalized coupling by $g_R = \Delta_R \kappa^{-2}$. Then, for a given initial distribution $P(\phi(0))$, we have

$$g_R = 1 - \int_{-\infty}^{\infty} \frac{P(u)}{1 + u^2\kappa^{-2}}\, du. \tag{91}$$

We may now define the renormalization group function $\beta(g_R) = \kappa(\partial g_R/\partial\kappa)$ in the usual way, and eliminate κ to express it in terms of g_R. In general, this cannot be done analytically, but it is straightforward to examine the neighbourhood of the fixed points.

For $\kappa \to \infty$, g_R approaches the ultraviolet fixed point $g_R = 0$. For small κ^{-2}, $g_R = O(\kappa^{-2})$, so that $\beta(g_R) = -2g_R + O(g_R^2)$, as expected. For $\kappa \to 0$, on the other hand $g_R \to 1$, and as may easily be shown, this is the infrared fixed point. In that limit, we may write

$$g_R = 1 - \kappa^2 \int_{-\infty}^{\infty} \frac{P(u) - P(0)}{\kappa^2 + u^2}\, du - \kappa^2 P(0) \int_{-\infty}^{\infty} \frac{du}{\kappa^2 + u^2} \tag{92}$$

so that $g_R = 1 + \pi\kappa P(0) + O(\kappa^2)$, and $\beta(g_R) = (g_R - 1) + O((g_R - 1)^2)$. Notice that the slope of the β-function at the non-trivial fixed point is universal as long as $P(u)$ is analytic at $u = 0$.

An interesting soluble case is when $P(u)$ is a Cauchy distribution, that is

$$P(u) = \frac{\lambda/\pi}{u^2 + \lambda^2}. \tag{93}$$

The integrals are now easy, and we find that $g_{\mathrm{R}} = \lambda/(\lambda + \kappa)$, which implies that

$$\beta(g_{\mathrm{R}}) = -g_{\mathrm{R}}(1 - g_{\mathrm{R}}) \tag{94}$$

the simplest possible β-function one could write down with the appropriate zeros. Note, however, that in this case the slope of the β-function at the origin is -1 rather than -2 as was found above. This may be traced to the fact that, for the Cauchy distribution, the second moment does not exist, and therefore the standard perturbation theory does not make sense. In fact, perturbation theory exists only when all moments of the initial distribution exist. Even in that case, the perturbation theory diverges. The integral in equation (91) may be seen as a kind of Borel summation of the expansion.

Nevertheless, the Cauchy distribution does give the correct universal behaviour near the infrared fixed point. This is because, at large t, in fact, for all $t > 0$, the distribution of $\phi(t)$ is bounded, and all moments exist. The case of $d = 0$ therefore gives an interesting solvable toy model in which the structure of the renormalization group may be studied.

6. Summary and further remarks

We have considered the evolution of a critical system from a random initial state, such as that which would follow a quench from high temperature, in the regime when the effects of thermal fluctuations may be neglected. Mathematically, this is described by a nonlinear diffusion equation with random initial conditions. We found that for $d \geqslant 2$ the nonlinearities are irrelevant, and the fluctuations at large times have a Gaussian distribution. For $d < 2$, there is a non-trivial fixed point within the ϵ expansion, and the fluctuations in local quantities like have a universal behaviour. The variance $\overline{\phi(x,t)^2}$ behaves like $1/t$, to all orders in ϵ, while the long time behaviour of the response function is governed by a non-trivial exponent γ_1^*. These results were confirmed both at large n, and by an exact calculation for $d = 0$. It is interesting to interpolate between the $\mathrm{O}(\epsilon^2)$ calculation and the exact results for $\epsilon = 2$. A simple Padé approach gives $\gamma_1^* \approx 0.53$ for $d = 1$ and $n = 1$. Similarly, for the universal amplitude of the local fluctuations, we find $A_2 \approx 0.15$ in this case.

The fact that, for $d < 2$, the fluctuations decay more rapidly than expected on the basis of the Gaussian model (which would give a $t^{-d/2}$ dependence for $\overline{\phi(x,t)^2}$), may be understood on the basis that the nonlinearities violate the conservation of the order parameter, and allow relaxation of fluctuations to progress more rapidly. A preliminary analysis of the case of a conserved order parameter (model B [1]) indicates that the nonlinearities do not accelerate the decay of local fluctuations.

It is somewhat disappointing that the upper critical dimension for this problem turns out to be two, as it renders the interesting part of the conclusions less applicable in physical situations. Nevertheless, we believe that the field theory we have constructed has a number of interesting features in its own right. In particular, it appears to have non-trivial behaviour for $d = 1$, unlike most equilibrium critical systems. If the model turns out to be soluble by some other method (as is suggested by its formal analogy to the nonlinear Schrödinger equation), this would give a unique example of a non-trivial field theory below its upper critical dimension.

In order to raise the upper critical dimension it is necessary to consider systems with other symmetries. It is not difficult to show that, in the case of a non-conserved order parameter, the upper critical dimension d_u for the relevance of disorder in the initial condition is always two less then the upper critical dimension for equilibrium fluctuations. Thus, a system described by a free energy with cubic interactions would have $d_u = 4$. Such an equation would describe the population dynamics of a simple birth–death process. In the realm of traditional critical phenomena, another example would be a spin glass, for which initial conditions are known to play an important role in the low temperature phase.

We conclude with some remarks on the case of a conserved order parameter. This is described by an equation of the type equation (3), with the replacement $\Gamma \to -\Gamma\nabla^2$. In addition, the fluctuations in the initial state must also respect the conservation of the total order parameter. This has the effect of replacing $\Delta \to -\Delta\nabla^2$. Such models have been considered extensively in the low temperature phase in the study of spinodal decomposition [9], and the subsequent dynamics of domain growth. In this context, the exactly soluble large n limit was also considered [12].

In the case of a quench down to the critical point (at the critical concentration), it is straightforward to set up the field theory formulation, as in this paper. Dimensional analysis then indicates that the coupling constant becomes dimensionless for $d = 0$. This appears not to be very interesting. It indicates that Gaussian fluctuations persist for all $d > 0$. An analysis of the soluble case of large n confirms this. However, it is also possible to consider a quench at a non-critical concentration. In this case, the relaxation time approaches infinity at the spinodal curve. Since the symmetry is now broken, the system is described by a model with cubic interactions, and the critical dimension is raised to two. However, an analysis of the corresponding field theory reveals the puzzling feature that, although the theory is non-renormalizable for $d > 2$, as expected (corresponding to the irrelevance of the interactions in the long time limit), exactly at $d = 2$ there appear to be no primitive divergences. This appears to make the kind of renormalization group program described in this paper difficult to carry out.

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Appendix A

We show how to estimate the strength Δ of the initial state randomness for the case of a quench from high temperatures. To be specific, we consider an Ising model with partition function

$$Z = \operatorname{Tr} \exp \left[\tfrac{1}{2}\beta \sum_{x,x'} J(x - x')s(x)s(x') + \beta H \sum_{x} s(x) \right] . \tag{A1}$$

In order to put this into continuum form, we use the standard transformation, writing the quadratic term in s as a Gaussian integral over a field ϕ, after which the trace over s may be performed. Finally, a gradient expansion is performed, and the naive continuum limit is taken. When this is done, keeping track of all of all the factors, we find that

$$Z = \int \mathcal{D}\phi \exp\left[-\int \frac{\mathrm{d}^d x}{a^d}\left((2\beta\widetilde{J})^{-1}[\phi^2 + R^2(\nabla\phi)^2] + \ln \mathrm{ch}(\phi + \beta H)\right)\right] \tag{A2}$$

where $\widetilde{J} = \sum_x J(x)$, and $R^2 = \widetilde{J}^{-1}\sum_x x^2 J(x)$ measures the range of the interaction. Equation (A2) is brought into the standard form of a Landau–Ginzburg free energy by expanding the exponent in powers of ϕ and rescaling $\phi \to (\beta\widetilde{J}a^d/R^2)^{1/2}\phi$. The result is a functional of the form

$$F = \int \left[\tfrac{1}{2}(\nabla\phi)^2 + (r/\phi)^2 + \tfrac{1}{4}\lambda\phi^4 + h\phi\right] \mathrm{d}^d x \tag{A3}$$

where

$$r = \frac{1-\beta\widetilde{J}}{R^2} \qquad \lambda \sim \frac{\beta^2\widetilde{J}^2 a^d}{R^4} \qquad h = \frac{(\beta\widetilde{J})^{1/2}}{Ra^{d/2}}\beta H. \tag{A4}$$

If we now identify this with the effective reduced free energy used for the dynamics, we see that Δ should be given by the $q = 0$ component of $\langle\phi(q)\phi(-q)\rangle$ at high temperature $\beta \to 0$. In that case, the nonlinear terms are negligible, and we see that that $\Delta \sim r^{-1} = R^2$.

Appendix B

We summarize the calculation of the various 2-loop Feynman integrals encountered.

The first is that of figure 2(*e*). It has a symmetry factor of 18, and is altogether

$$18(2\pi\Delta)^2 \frac{1}{-\mathrm{i}\omega + q^2} \times \int \frac{\mathrm{d}k_1 \mathrm{d}k_2}{(-\mathrm{i}\omega + k_1^2 + k_2^2 + (q + k_1 + k_2)^2)(-\mathrm{i}\omega + q^2 + 2k_1^2 + 2k_2^2)} \tag{B1}$$

which is to be evaluated at $\omega = 0$, $q = \kappa$. Using the standard Feynman parameter method, this becomes

$$18(2\pi\Delta)^2\kappa^{-2}\int_0^1 \mathrm{d}x \times \int \frac{\mathrm{d}k_1 \mathrm{d}k_2}{(2k_1^2 + 2k_2^2 + 2xk_1\cdot k_2 + 2xq\cdot k_1 + 2xq\cdot k_2 + (1-x)q^2)^2}. \tag{B2}$$

Although it is possible to evaluate this directly in $2-\epsilon$ dimensions, in order to extract the residue of the simple pole at $\epsilon = 0$, which comes from the large k behaviour, we may neglect the q-dependence in the integrand. Defining $k_\pm = k_1 \pm k_2$, the integral becomes

$$\frac{1}{4}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}k_+ \mathrm{d}k_-}{((1+\frac{1}{2}x)k_+^2 + (1-\frac{1}{2}x)k_-^2)^2} \tag{B3}$$

which, with a cutoff, would behave for $d = 2$ like

$$\tfrac{1}{4}2\pi^2 \ln \Lambda \int_0^1 \frac{\mathrm{d}x}{(1+\frac{1}{2}x)(1-\frac{1}{2}x)}. \tag{B4}$$

In dimensional regularization, the factor Δ^2 must accompany a factor of $\kappa^{-2\epsilon}$, and therefore $\ln \Lambda$ is replaced by $\kappa^{-2\epsilon}/2\epsilon$. The integral over x gives $\ln 3$, and putting these factors together we find the result quoted in equation (32).

The contribution of figure 4(f) to $G_{\phi\tilde{\phi}}$ is

$$6(2\pi\Delta)^2\kappa^{-4} \int \frac{\mathrm{d}k_1 \mathrm{d}k_2}{(k_1^2 + k_2^2 + (q + k_1 + k_2)^2)^2} \tag{B5}$$

in which the integral, by the same substitution, becomes

$$\frac{1}{4} \int \frac{\mathrm{d}k_+ \mathrm{d}k_-}{(\frac{3}{2}k_+^2 + \frac{1}{2}k_-^2)^2} \sim \frac{1}{3} 2\pi^2 \ln \Lambda \tag{B6}$$

giving the result for I_f in equation (34).

The contribution of figure 6, which represents a dressed insertion of the operator $(\nabla\tilde{\phi})^2$, is

$$\begin{aligned} 2(2\pi\Delta)^2 &\int \frac{(k_1 + \frac{1}{3}q)^2 \delta(k_1 + k_2 + k_3)\mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3}{\left((k_1 + \frac{1}{3}q)^2 + (k_2 + \frac{1}{3}q)^2 + (k_3 + \frac{1}{3}q)^2\right)^2} \\ &= \frac{2(2\pi\Delta)^2}{3} \int \frac{\delta(k_1 + k_2 + k_3)\mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3}{k_1^2 + k_2^2 + k_3^2 + \frac{1}{3}q^2}. \end{aligned} \tag{B7}$$

To evaluate the renormalization of $\tilde{\Delta}_2$ we need the derivative with respect to q^2. This gives

$$-\frac{2(2\pi\Delta)^2}{9} \int \frac{\mathrm{d}k_1 \mathrm{d}k_2}{\left(k_1^2 + k_2^2 + (k_1 + k_2)^2\right)^2} \tag{B8}$$

which may be evaluated by similar techniques to give the result quoted in equation (75).

Finally, the first non-trivial contribution to the amplitude A_4 (which is, in fact, at 3-loop order) is shown in figure 5. This is more easily evaluated in the (q, t) representation

$$-24(2\pi\Delta)^3 \int_0^t \mathrm{d}t' \int \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{e}^{-(t+t')(k_1^2+k_2^2+k_3^2)} \mathrm{e}^{-(t-t')(k_1+k_2+k_3)^2}. \tag{B9}$$

By the rules of Gaussian integration, this is

$$-24(2\pi\Delta)^3 \frac{\pi^3}{(2\pi)^6} \int_0^t D^{-1}\,\mathrm{d}t' \tag{B10}$$

where, to the required accuracy, we have taken $d = 2$, and

$$D = \begin{vmatrix} 2t & t-t' & t-t' \\ t-t' & 2t & t-t' \\ t-t' & t-t' & 2t \end{vmatrix} = 2(2t-t')(t+t')^2 \tag{B11}$$

giving the final answer

$$-\frac{24(2\pi\Delta)^3}{2t^2}\left(\frac{3+4\ln 2}{18}\right). \tag{B12}$$

The result in equation (65) then follows on inserting the fixed point value for Δ.

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